Exercise 5 Stochastic Models of Manufacturing Systems 4T400, 11 March

1. Jobs arrive at two parallel machines, each with its own buffer, according to a Poisson stream with a rate of 10 jobs per hour. The processing times are exponential with a mean of 4 minutes on machine 1 and 8 mintues on machine 2. On arrival jobs are assigned with equal probability to the buffer of machine 1 or 2.

   (a) Determine the mean flow time (waiting time plus processing time) of a job sent to machine 1, sent to machine 2, and also of an arbitrary job.

   (b) Determine the fraction of jobs with a flow time longer than 30 minutes.

   (c) Suppose that arriving jobs are sent with probability p to machine 1 and with probability $1-p$ to machine 2. For which p is the mean flow time of an arbitrary job minimal?

   Answer:

   (a) For machine 1 we have $\lambda_1 = \frac{1}{12}$ (jobs/min) and $\frac{1}{\mu_1} = 4$ (min), so $\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{1}{3}$ and

   $$E(S_1) = \frac{\frac{1}{\mu_1}}{1-\rho_1} = 6 \text{ (min)}.$$

   Similarly, for machine 2 we have $\lambda_2 = \frac{1}{12}$ (jobs/min), $\frac{1}{\mu_2} = 8$ (min), so $\rho_2 = \frac{\lambda_2}{\mu_2} = \frac{2}{3}$ and

   $$E(S_2) = \frac{\frac{1}{\mu_2}}{1-\rho_2} = 24 \text{ (min)}.$$

   For the mean flow time of an arbitrary job we get

   $$E(S) = \frac{1}{2}E(S_1) + \frac{1}{2}E(S_2) = 15 \text{ (min)}.$$

   (b) For a job on machine 11 we have

   $$P(S_1 > 30) = e^{-\mu_1(1-\rho_1)30} = e^{-5} \approx 0.0067,$$

   and for a job on machine 2,

   $$P(S_2 > 30) = e^{-\mu_2(1-\rho_2)30} = e^{-5/4} \approx 0.287.$$

   Hence, for an arbitrary job,

   $$P(S > 30) = \frac{1}{2}P(S_1 > 30) + \frac{1}{2}P(S_2 > 30) = \frac{1}{2}e^{-5} + \frac{1}{2}e^{-5/4} \approx 0.147.$$

   (c) The mean flow time of an arbitrary job is equal to

   $$\begin{aligned} E(S(p)) &= \frac{p\frac{1}{\mu_1}}{1-p\frac{\lambda}{\mu_1}} + \frac{(1-p)\frac{1}{\mu_2}}{1-(1-p)\frac{\lambda}{\mu_2}} \\ &= \frac{12p}{3-2p} + \frac{24-24p}{4p-1} \\ &= \frac{18}{3-2p} + \frac{18}{4p-1} - 12. \end{aligned}$$

   The routing probability p for which the mean flow time is mimimized, satisfies

   $$\frac{d}{dp}E(S(p)) = \frac{36}{(3-2p)^2} - \frac{72}{(4p-1)^2} = 0,$$

so

$$\frac{1}{3-2p} = \frac{\sqrt{2}}{4p-1},$$

and thus the optimal p is given by

$$p = \frac{1}{4} \cdot \frac{6+\sqrt{2}}{1+\sqrt{2}} \approx 0.768.$$

2. One has to decide between two machines, machine 1 and 2, for the processing of jobs. The mean processing time of machine 1 is 2 minutes and the standard deviation is 6 minutes. Machine 2 is slower: the mean processing time is 3 minutes. On the other hand, machine 2 is also more reliable: the standard deviation is only 1 minute. Suppose jobs arrive according to a Poisson process. Plot for both machines the mean flow time as a function of the arrival rate. Which machine do you prefer?

   Answer: The mean flow time is equal to

   $$E(S) = \frac{\rho}{1-\rho}\frac{E(B)}{2}\left(1+c_B^2\right) + E(B),$$

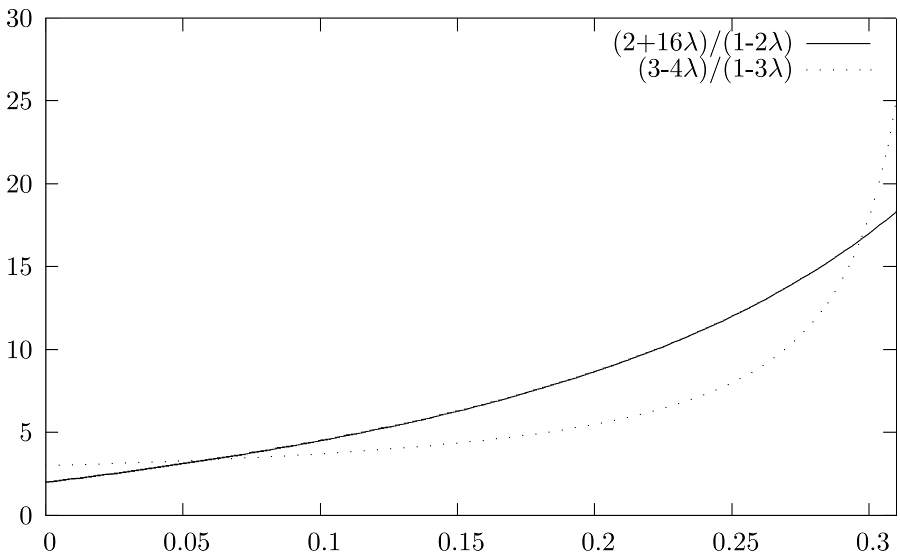
   where $E(B)$ is the mean processing time, c_B the coefficient of variation of the processsing time, $\rho = \lambda E(B)$ the utilization and λ the arrival rate. For machine 1 we have $E(B) = 2$ minutes and $c_B = 3$, so $E(S) = 12$ minutes, and thus

   $$E(S_1) = \frac{2+16\lambda}{1-2\lambda}.$$

   For machine 2 we have $E(B) = 3$ minutes and $c_B = 1/3$, so $E(S) = 8$ minutes, and thus

   $$E(S_2) = \frac{3-4\lambda}{1-3\lambda}.$$

   The figure below shows the mean flow time as a function of λ for both machines.



   The conclusion is that it depends on the load whether the slower, reliable machine performs better than the faster, unreliable machine.

3. Consider a machine where jobs arrive according to a Poisson process with rate λ jobs per hour. The processing times are exponential with rate μ jobs per hours (with $\lambda < \mu$). Calculate the fraction of jobs for which the flow time is greater than a times the mean flow time.

   Answer: The sojourn time distribution is exponential with mean $\frac{1}{\mu(1-\rho)}$, where $\rho = \frac{\lambda}{\mu}$. Hence,

$$P(S > \frac{a}{\mu(1-\rho)}) = e^{-\mu(1-\rho)\frac{a}{\mu(1-\rho)}} = e^{-a}.$$

4. In a dairy barn, cows arrive at a milking robot according to a Poisson stream with a rate of 6.5 cows per hour. The cows are milked by the robot in order of arrival. Data (in minutes) on the milking times is available.

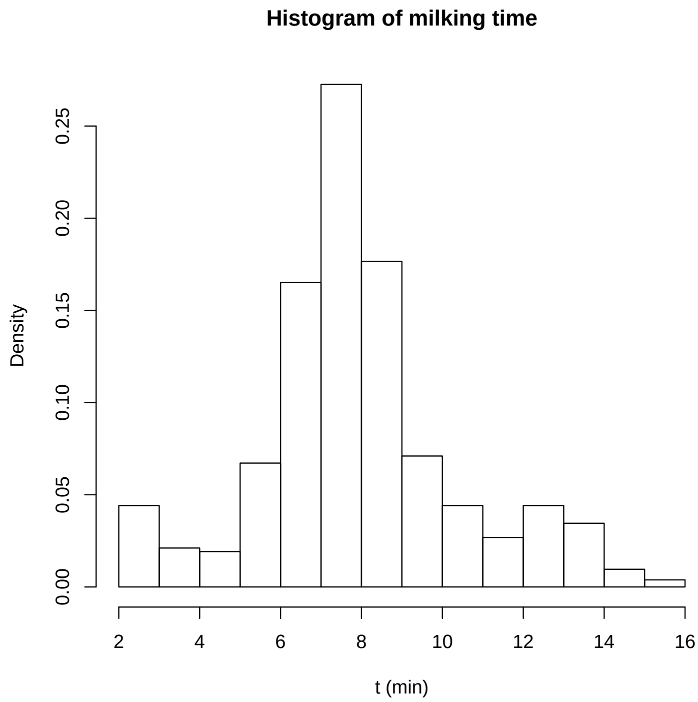
   (a) Calculate the sample mean and sample standard deviation of the collected milking times. Plot also a histogram of the milking times.

   (b) Estimate the mean flow time (waiting time plus milking time) of a cow, based on the sample mean and sample standard deviation of the milking times.

   (c) Plot a histogram of the flow time distribution obtained by simulation (over a sufficiently long period, using the historical data on milking times). Compare this distribution with the exponential distribution with the same mean. What is your conclusion?

   Answer:

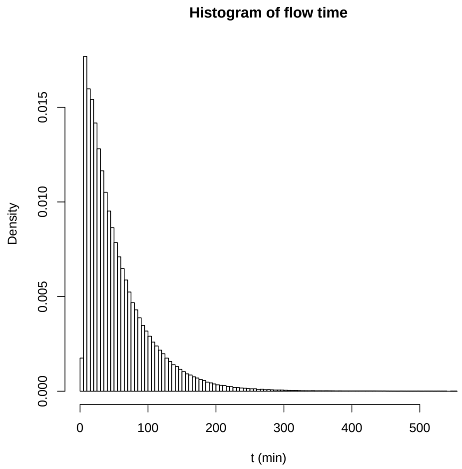
   (a) The sample mean is 8.41 minutes, the sample standard deviation is 2.53 minutes, and thus the coefficient of variation is 0.30. The histogram of the milking times is listed below.

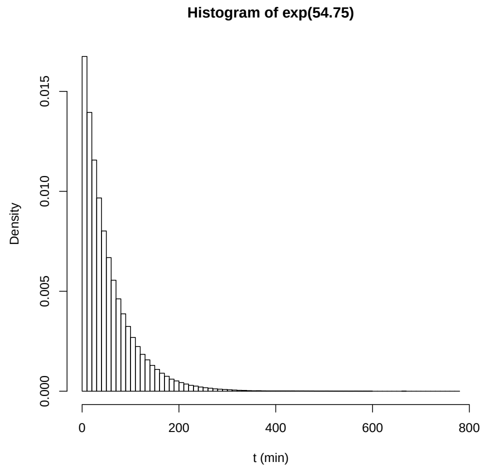


   (b) WWe have $\lambda = 6.5/60$ cows per minute, so $\rho = \lambda E(B) = 0.911$. Hence,

$$E(S) = \frac{\rho}{1-\rho}\frac{E(B)}{2}(1 + c_B^2) + E(B) = 54.75 \text{ (min)}.$$

(c) Below is a histogram of the flow time distribution (left graph) obtained by a χ simulation program (for 10^6 flow time realizations), using the historical data on milking times, together with a histogram of 10^6 samples from the exponential distribution with mean $E(S) = 54$ minutes (right graph).





Comparison of both histograms reveals that the sojourn time distribution is very close to an exponential, which is no surprise, since the utilization is high, i.e., $\rho = 0.911$.

5. A miniload is a compact storage system for totes, where an automated crane, which can simultaneously move fast in horizontal and vertical direction, brings the required tote from the shelve to the orderpicker (see Fig. 1). The speed of the crane in the horizontal direction is 2,5 m/s and the vertical speed is 0,4 m/s. The storage rack with totes is schematically depicted in Fig. 2. The length of the rack is 25 m and its height is 4 m. The horizontal location of a tote, that has to be picked, is uniform between 0 and 25 m, and it is independent of its vertical location, which is uniform between 0 and 4 m. The crane always starts at the left corner below (drop-off location), and returns to that location with the required tote. The random variable X denotes the travel time of the crane to the required tote in the horizontal direction, and Y is the travel time in the vertical direction.

Figure 1: Miniload.

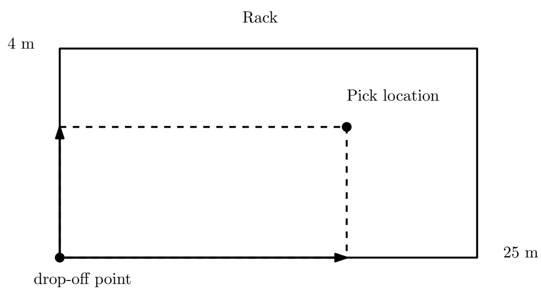


Figure 2: Rack with totes.

(a) Motivate why X and Y are both uniform between 0 and 10 seconds.

(b) Since the crane can simultaneously move in horizontal and vertical direction, the travel time Z of the crane to the tote is equal to $Z = \max(X, Y)$. Show that the distribution function $F(t) = P(Z \leq t)$ of the travel time is given by

$$F(t) = P(Z \leq t) = \begin{cases} 1 & t > 10 \\ \frac{t^2}{100} & 0 < t \leq 10 \\ 0 & t \leq 0 \end{cases}$$

(c) Calculate the density $f(t)$ of the travel time.

(d) Calculate the mean and variance of the travel time Z.

(e) It takes the crane exactly 5 seconds to fetch a tote from the rack. When the crane returns to the drop-off location, it puts the tote on a roller conveyor (which will transport the tote to the orderpicker). The time to put the tote on the roller conveyor also takes 5 seconds. Hence, the total time to collect a tote from the rack consists of the travel time of the crane (back and forth), the time to fetch the tote from the rack and to the time to put it on the roller conveyor. Calculate the capacity of the crane, that is, the mean number of totes that the crane can retrieve from the rack per hour.

(f) Orders for totes arrive at the crane according to a Poisson stream with a rate of 60 orders per hour. Each order requires one tote. Calculate the mean throughput time of an order (waiting time plus time to collect the tote).

(g) Calculate the mean number of orders (waiting or in process) at the crane.

Answer:

(a) The horizontal location is uniform on $(0, 25)$ m and the speed is 2,5 m/s, so the travel time is uniform on $(0, 10)$ seconds. Same for vertical travel time.

(b) For Z we have

$$\begin{aligned} F(t) &= P(Z \le t) = P(\max(X, Y) \le t) = P(X \le t, Y \le t) \\ &= P(X \le t)P(Y \le t) = \frac{t}{10}\frac{t}{10} = \frac{t^2}{100}, \quad 0 < t \le 10. \end{aligned}$$

(c)

$$f(t) = \frac{d}{dt}F(t) = \frac{t}{50}, \quad 0 < t \le 10,$$

and $f(t) = 0$ for $t < 0$ and $t > 10$.

(d)

$$\begin{aligned} E(Z) &= \int_0^{10} \frac{t^2}{50} dt = 6\frac{2}{3} \text{ (sec)}, \\ E(Z^2) &= \int_0^{10} \frac{t^3}{50} dt = 50 \text{ (sec}^2\text{)}, \\ \text{var}(Z) &= E(Z^2) - (E(Z))^2 = \frac{50}{9} = 5.55 \text{ (sec}^2\text{)} \end{aligned}$$

(e) The mean total time to collect a tote is $E(B) = 5 + 2E(Z) + 5 = 23\frac{1}{3}$ seconds. Hence the capacity is $3600/23\frac{1}{3} = 154$ totes per hour.

(f) The mean total time to collect a tote is $E(B) = 23\frac{1}{3}$ seconds, the variance is $\text{var}(B) = \text{var}(2Z) = 4\text{var}(Z)$, so the squared coefficient of variation is $c_B^2 = 4\text{var}(Z)/t_e^2 = 2/49$. The (Poisson) arrival rate of orders is $\frac{1}{60}$ orders per second. The utilization ρ is this equal to $\frac{E(B)}{60} = 0.39$. For the mean throughput time $E(S)$ we then get

$$E(S) = \frac{\rho}{1-\rho}\frac{1}{2}E(B)(1 + c_B^2) + E(B) = 31.1 \text{ seconds.}$$

(g) Little's law yields $E(L) = \frac{1}{60} \cdot 31.1 = 0.52$ orders.